Phase synchronization of coupled Ginzburg-Landau equations

Lutz Junge* and Ulrich Parlitz
Drittes Physikalisches Institut, Universität Göttingen, Bürgerstraße 42-44, D-37073 Göttingen, Germany
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The occurrence of phase synchronization of a pair of unidirectionally coupled nonidentical Ginzburg-Landau equations is demonstrated and characterized using cyclic and extended phases. Furthermore, it is shown that weak coupling first leads to frequency synchronization and later to phase synchronization. For strong coupling there is evidence for generalized synchronization.

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Synchronization of periodic signals is a well-known phenomenon in science and engineering. However, even chaotic systems may be linked in a way such that their chaotic oscillations are synchronized [1–3]. If a pair of very similar or even identical systems is coupled one may observe *identical synchronization* where the difference of the state vectors of both systems converges to zero, even in the case of chaotic dynamics. Such synchronization phenomena may not only be observed for low-dimensional systems but also for spatially extended dynamics [4–6]. Identical synchronization, however, can only occur for pairs of identical systems but not for coupled systems that are of completely different origin (e.g., an electrical circuit coupled to a mechanical system). What does ‘‘synchronization’’ mean in such a more general case? Periodic systems are usually called synchronized if either their phases or frequencies are locked. For chaotic systems, however, the notions of ‘‘frequency’’ or ‘‘phase’’ are in general not well defined, except for some class of chaotic systems where a phase variable can be introduced and chaotic *phase synchronization* (PS) may be observed [7]. Since (chaotic) PS turned out to be a rather robust phenomenon it was also observed with binary coupling [8] and in noisy environments like magnetoencephalography measurements [10]. In data analysis (statistical) evidence for PS is used to evaluate possible coupling between different physical processes like, for example, the solar activity cycle (sunspot numbers) with the solar inertial motion [11]. Recently PS of spatiotemporal chaos under harmonic forcing was observed and analyzed in Chaté *et al.* [12]. In this paper we demonstrate that PS may also occur with spatiotemporal chaos of coupled nonidentical partial differential equations (PDEs). As an example we have chosen a pair of unidirectionally coupled Ginzburg-Landau equations (GLEs),

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= u-(1-i\alpha)\frac{\partial^2 u}{\partial x^2}+(1+i\beta_1)|u|^2 u, \quad x\in[0,L],\\
\frac{\partial v}{\partial t} &= v-(1-i\alpha)\frac{\partial^2 v}{\partial x^2}+(1+i\beta_2)|v|^2 v + c(u-v),
\end{aligned}
\tag{1}
$$

with periodic boundary conditions. The GLE is a fundamental model for structure formation [13]. Depending on the parameters α and β *defect turbulence* or *phase turbulence* of the complex variable u may occur resulting in spatiotemporal chaotic dynamics. In our investigations the parameters of the driving system $\alpha=2.0$ and $\beta_1=0.7$ are chosen to be in the phase turbulent regime [see Fig. 1(a)]. In order to study nonidentical systems we have chosen for the response system the same α value but different β values. $\beta_2=0.9$ results in more turbulent phase dynamics and $\beta_2=1.05$ leads to defect turbulence [see Fig. 1(b)]. All calculations were performed for a fixed system length $L=100$. The PDE was solved using an implicit scheme that is second order in space and first order in time.

To examine PS one has to find a suitable quantity that represents a phase of the system and that usually corresponds to a zero Lyapunov exponent. Even in low-dimensional systems this is often a nontrivial problem. The advantage of using the GLE is that we have a complex variable that allows us to use polar coordinates with a unique phase definition $\phi(x,t)\in\mathbb{R}$. The coupled systems (1) are regarded as phase synchronized [10] if their phase difference is bounded from above [14]:

$$
|\phi_u(x,t)-\phi_v(x,t)|<\text{const}; \quad \forall x\in[0,L], \forall t>T, \tag{2}
$$

where T denotes the transient time. The mean frequency of the system can be defined as

$$
\Omega=\lim_{t\to\infty}\frac{\langle\phi(x,t)\rangle_x}{t}, \tag{3}
$$

where $\langle\,\rangle_x$ is the spatial average. PS implies that the frequency mismatch

$$
\Delta\Omega=\Omega_u-\Omega_v=0 \tag{4}
$$

of the mean frequencies Ω_u and Ω_v vanishes and frequency synchronization (FS) occurs. We want to stress that the opposite is in general not true because of possible (rare) phase slips where the relative phases change rapidly by $\pm 2\pi$. When a coupling parameter of the systems is changed one often observes a scenario where first the mean frequencies synchronize (FS) and later PS occurs (i.e., all phase slips disappeared). Note that we consider here extended phases $\phi(x,t)$ on the whole real axis $\mathbb{R}$ for diagnosing PS, because we are interested in the long term evolution of the phase difference.

*Electronic address: junge@dpi.physik.uni-goettingen.de

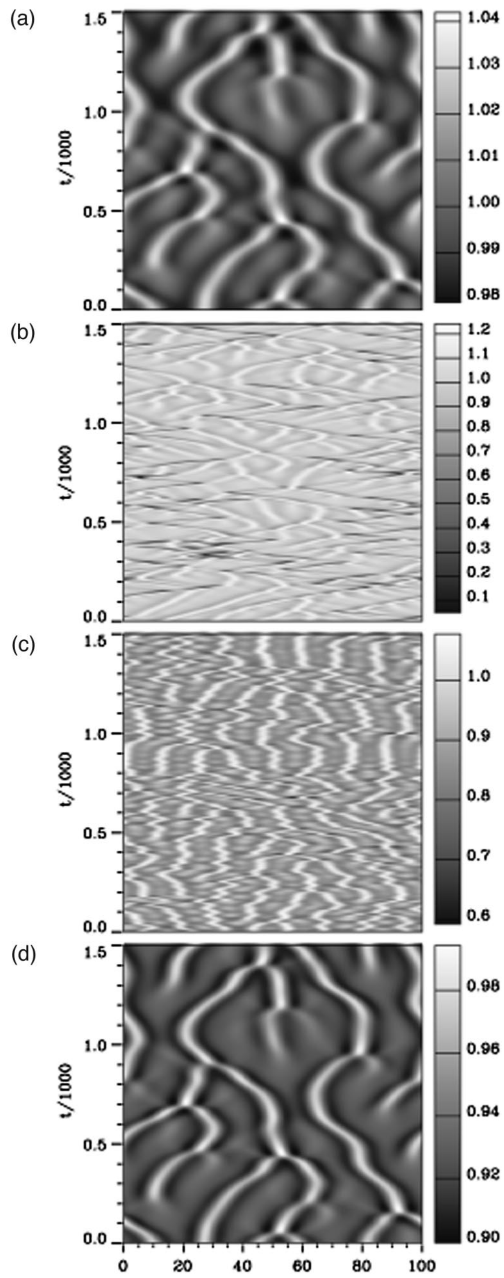

FIG. 1. Amplitude dynamics of the coupled Ginzburg-Landau equations: (a) drive $\beta_1 = 0.7$, (b) response system $\beta_2 = 1.05$ without coupling, (c) weak coupling of $c = 0.11$, beginning of phase synchronization, (d) strong coupling of $c = 0.2$, strong correlation of the amplitude patterns.

To have a quantitative measure for PS in spatially extended systems we compute the maximum relative phase difference of the extended phases $\phi_{u,v}$,

$$\Delta\phi = \max_{x \in L, T \leq t \in \mathbb{R}} |\phi_u(x,t) - \phi_v(x,t)|. \tag{5}$$

After a transient time T, we set the initial phase difference to $|\phi_u(x,0) - \phi_v(x,0)| \leq \pi$ [9] and with perfect PS the phase difference $\Delta\phi$ will not exceed the bound of 2π.

Figure 2 shows the maximum relative phase difference $\Delta\phi$ in dependence on the coupling strength c after an evolution of 2500 time units for $\beta_2 = 1.05$ (solid curve) and $\beta_2 = 0.9$ (dashed curve). The dotted line shows the bound 2π for PS. The phase difference decreases continuously for increasing coupling strength c. For the case $\beta_2 = 0.9$ (phase turbulence) a transition to PS occurs at $c = 0.07$ and with $\beta_2 = 1.05$ (defect turbulence) the onset of PS is at $c = 0.11$. For visualization Fig. 1 shows the evolution of the amplitude dynamics of (a) the driving system $\beta_1 = 0.7$, (b) the response $\beta_2 = 1.05$ without coupling, (c) at the onset of PS, and (d) for strong coupling. Note, that in Fig. 1(c) both PDEs are phase synchronized while the amplitudes are totally uncorrelated. For strong coupling, see Fig. 1(d), even the amplitude patterns become similar to each other.

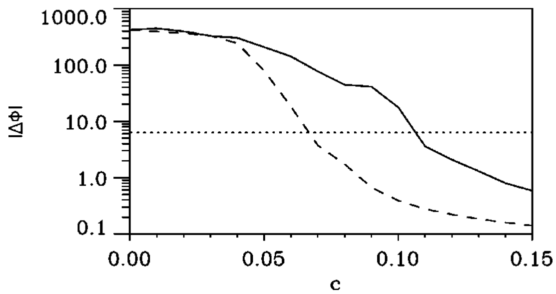

FIG. 2. Relative phase difference (5) vs coupling strength $c = 0.0, \ldots, 0.15$. The solid line corresponds to $\beta_2 = 1.05$ (defect turbulence), the dashed curve to $\beta_2 = 0.9$ (phase turbulence). Below the dotted line the phase difference is less than 2π, which corresponds to phase synchronization.

Figure 3 shows the average frequency mismatch Eq. (3) against the coupling strength c. As expected PS implies FS and we observe a locking of the frequencies before perfect phase locking is observed:

$$\begin{aligned} \beta_2 = 0.90: &\quad c \geq 0.06 \Rightarrow \text{FS}, \quad c \geq 0.07 \Rightarrow \text{PS}, \\ \beta_2 = 1.05: &\quad c \geq 0.10 \Rightarrow \text{FS}, \quad c \geq 0.11 \Rightarrow \text{PS}. \end{aligned} \tag{6}$$

In this intermediate regime we observed phase slips that belong mainly to defects, but they occur so seldom that they have a negligible effect on the mean frequency Ω_2.

An alternative way for detecting phase synchronization [10,11] is to look for a pronounced peak in the distribution of the relative cyclic phase difference,

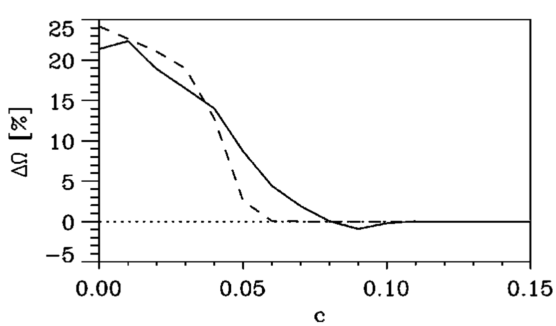

FIG. 3. Relative frequency difference $\Delta\Omega$ vs coupling strength $c = 0.0, \ldots, 0.15$. The solid line corresponds to $\beta_2 = 1.05$, the dashed curve to $\beta_2 = 0.9$.

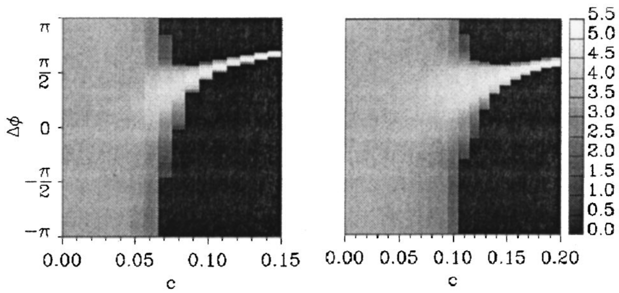

FIG. 4. Histogram of the relative cyclic phase difference (7) vs coupling strength c. The left plot corresponds to $\beta_2=0.9$ and the right one to $\beta_2=1.05$. The logarithm of the bin content is plotted in gray scale.

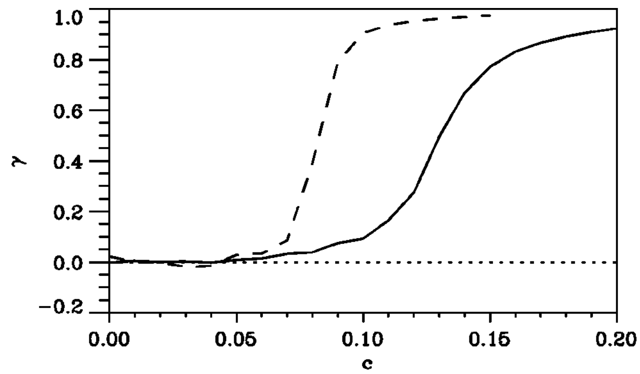

FIG. 5. Linear correlation coefficient vs coupling strength $c=0.0,\dots,0.15$. The solid line corresponds to $\beta_2=1.05$, the dashed to $\beta_2=0.9$.

$$\widetilde{\Delta\phi}=\widetilde{\phi_u}(x,t)-\widetilde{\phi_v}(x,t)\in[-\pi,\pi], \qquad (7)$$

with $\widetilde{\phi_i}=\phi_i \bmod 2\pi\in S^1$ (unit circle) [15]. For FS it is sufficient to observe a sharp peak in the phase distribution, which implies that the systems have at almost all time the same mean frequency. Additionaly, to detect perfect PS (without phase slips) it is necessary that the histogram of the cyclic phase differences has support on a proper subset on the circle with diameter $d(\{\widetilde{\Delta\phi}\})<2\pi$, i.e., a gap in the histogram appears and phase slips no longer occur [16]. Figure 4 shows the histogram of the cyclic phase differences for the two examined regimes. We have plotted the logarithm of the bin contents to be able to observe the first occurrence of the gap. Without coupling $\widetilde{\Delta\phi}$ is distributed equally on $[-\pi,\pi]$. For small coupling c a (not very sharp) maximum in the histogram appears showing the tendency of the systems for synchronizing their phases/frequencies (not clearly visible with gray scaling used in Fig. 4). At the onset of FS the histogram becomes sharper but covers still the whole circle $[-\pi,\pi]$, which indicates the occurrence of (rare) phase slips. A gap appears in the left plot ($\beta_2=0.9$) for $c=0.07$ and in the right ($\beta_2=1.05$) for $c=0.11$, exactly at the onset of PS, see Fig. 2 and Eq. (6). Further increasing of the coupling strength c leads to a histogram with a delta peaklike function around a fixed value.

Now we want to examine how similar the amplitude patterns $|u|,|v|$ are in the phase synchronized regime.

To compare the two patterns quantitatively we have calculated the linear product-moment correlation coefficient γ (or *Pearson's r*)

$$\gamma=\frac{\sum_i (|u_i|-\overline{u})(|v_i|-\overline{v})}{\sqrt{\sum_i (|u_i|-\overline{u})^2}\sqrt{\sum_i (|v_i|-\overline{v})^2}}, \qquad (8)$$

where $\overline{u},\overline{v}$ are the mean values of $|u|,|v|$, respectively and $-1\leq\gamma\leq1$. A value near 0 indicates that the data are linearly uncorrelated; for $\gamma=1$ we have complete positive correlation and for $\gamma=-1$ the data are negatively correlated. Figure 5 shows the (linear) correlation between drive and response pattern against the coupling strength c for $\beta_2=1.05$ (solid line) and $\beta_2=0.9$ (dashed line). At the onset of PS at $c=0.11$ (0.07) the amplitude patterns are totally uncorrelated, see Fig. 1(c). Increasing the coupling strength induces stronger correlations and for strong coupling the dynamics become very similar, but not equal because the systems are nonidentical and therefore a synchronization manifold $u\equiv v$ does not exist. While due to the parameter mismatch $\beta_1\neq\beta_2$ identical synchronization is not possible; a functional relationship between u and v may exist, i.e., *generalized synchronization* (GS) may occur [17]. A necessary condition for GS is that all tranversal Lyapunov exponents are negative $\lambda_\perp^{max}<0$ [18]. To check this we computed the largest transversal Lyapunov exponents of the driven system for several coupling strengths c and both examined param-

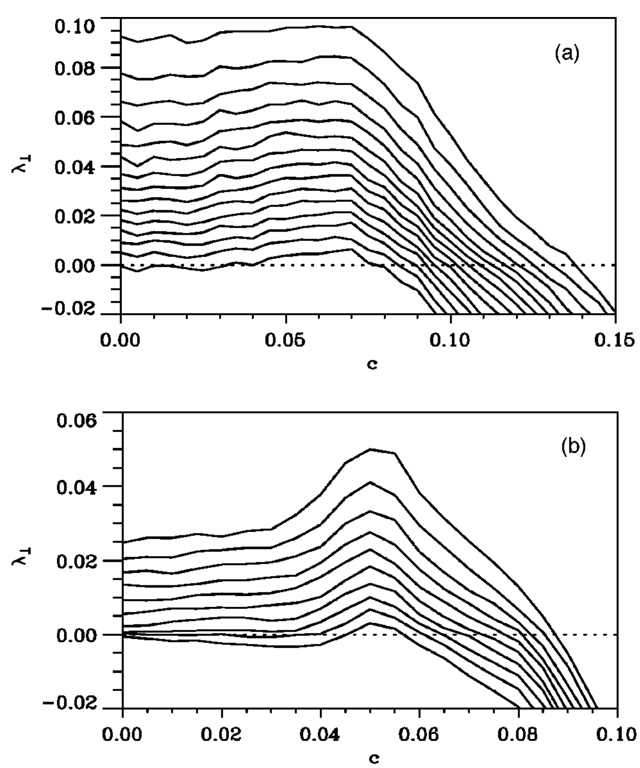

FIG. 6. The largest transversal Lyapunov exponents $\lambda_\perp$ vs coupling strength c. (a) $\beta_2=1.05$, $c=0.0,\dots,0.15$, (b) $\beta2=0.9$, $c=0.0,\dots,0.1$.

eter sets. Figures 6(a,b) show the spectrum of the driven GLE (1) for $\beta_2=1.05$ ($\beta_2=0.9$). At the onset of PS for $c=0.11$ ($c=0.07$) we have still six (six) positive Lyapunov exponents and no GS. For $c=0.15$ ($c=0.09$) all transversal Lyapunov exponents are negative indicating GS. This transition is accompanied by a sharp increase of the linear correlation coefficient γ, which is another indication of a fixed relation between the flows.

In this paper we have demonstrated the occurrence of spatiotemporal phase synchronization in a system consisting of two unidirectionally coupled Ginzburg-Landau equations (1). Above a certain threshold value of the coupling parameter c the frequencies $\Omega_{u,v}$ synchronize while rare phase slips are still observed. Slightly above this threshold the relative phase differences $\Delta\phi$ also remain bounded and PS without any phase slips occurs. Another test for PS, the histogram of the relative cyclic phase differences, confirms the results. For strong coupling all transversal Lyapunov exponents are negative and the patterns are highly correlated, which indicates the presence of generalized synchronization.

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[1] V.S. Afraimovich, N.N. Verichev, and M.I. Rabinovich, Radiophys. Quantum Electron. **29**, 1050 (1986).

[2] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); L. Pecora and T. Carroll, Phys. Rev. Lett. **64**, 821 (1990); L. Kocarev and U. Parlitz, *ibid.* **74**, 5028 (1995); U. Parlitz, L. Kocarev, T. Stojanovski, and H. Preckel, Phys. Rev. E **53**, 4351 (1996).

[3] U. Parlitz, L. Kocarev, T. Stojanovski, and L. Junge, Physica D **9**, 139 (1997).

[4] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **77**, 2206 (1996); L. Kocarev, Z. Tasev, and U. Parlitz, *ibid.* **79**, 51 (1997); N. Parekh, S. Parthasarathy, and S. Sinha, *ibid.* **81**, 1401 (1998); L. Junge, U. Parlitz, Z. Tasev, and L. Kocarev, Int. J. Bifurcation Chaos Appl. Sci. Eng. **9**, 2265 (2000); L. Junge and U. Parlitz, Phys. Rev. E **61**, 3736 (2000).

[5] G. Hu and Z. Qu, Phys. Rev. Lett. **72**, 68 (1994); R.O. Grigoriev, M.C. Cross, and H.G. Schuster, *ibid.* **79**, 2795 (1997).

[6] S. Boccaletti, J. Bragard, F.T. Arecchi, and H. Mancini, Phys. Rev. Lett. **83**, 536 (1999).

[7] M.G. Rosenblum, A. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996); U. Parlitz, L. Junge, W. Lauterborn, and L. Kocarev, Phys. Rev. E **54**, 2115 (1996); G. Osipov, A. Pikovsky, M. Rosenblum, and J. Kurths, *ibid.* **55**, 2353 (1997); M.G. Rosenblum, A.S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).

[8] U. Parlitz and I. Wedekind, Int. J. Bifurcation Chaos (to be published).

[9] We can set the initial phase difference $|\Delta\phi|<\pi$ because the initial phases are only determined up to multiples of 2π.

[10] P. Tass, M. Rosenblum, J. Weule, J. Kurths, A. Pikovsky, J. Volkmann, A. Schnitzler, and H.-J. Freund, Phys. Rev. Lett. **81**, 3291 (1998); C. Schäfer, M.G. Rosenblum, J. Kurths, and H.-H. Abel, Nature (London) **392**, 239 (1998).

[11] M. Palus, J. Kurths, U. Schwarz, D. Novotna, and I. Charvatova, Int. J. Bifurcation Chaos (to be published).

[12] H. Chaté, A. Pikovsky, and O. Rudzick, Physica D **131**, 17 (1999).

[13] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[14] The most general definition considers also rational relations between the phases and frequencies $\Delta\phi=|n\phi_u-m\phi_v|<\mathrm{const}\Rightarrow\Delta\Omega=n\Omega_1-m\Omega_2=0$; $n,m\in\mathbb{N}$, which are the classical $n{:}m$ resonances.

[15] Recently Boccaletti *et al.* [6] have investigated synchronization features of bi-directionally coupled nonidentical GLEs. As indicators for PS these authors used the convergence of the averaged phase differences $\widetilde{\Delta\psi}=\langle|\widetilde{\phi_u}-\widetilde{\phi_v}|\rangle$ to a constant value when varying the coupling strength c. A motivation for this measure of PS is not given in Ref. [6] and we consider it not to be suitable for verifying PS or FS.

[16] This argument applies only in the noise-free case. In experiments the (usually Gaussian) noise induces phase slips and support on a proper subset of the circle is no longer possible. Therefore, with noisy data PS can only be verified in a statistical sense by observing a sharp peak in the histogram Fig. 4.

[17] N. Rulkov, M. Sushchik, L. Tsimring, and H. Abarbanel, Phys. Rev. E **51**, 980 (1995); L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).

[18] $\lambda_\perp<0$ is not a *sufficient* condition, because there may exist some transversely unstable sets that can locally destroy the functional relationship.